Math 429 - Exercise Sheet 11

1. Explicitly write down the Cartan matrices corresponding to $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ (based on the simple roots you worked out last time) and check that the corresponding Dynkin diagrams are indeed B_n, C_n, D_n.

Solution. Consider the root system D_n associated to the Lie algebra $\mathfrak{o}_{2n}$. Recall our choice of simple roots

$$\beta_k = e_k - e_{k+1},\ k = 1, \ldots, n-1, \quad \beta_n = e_{n-1} + e_n.$$

from Exercise Sheet 9. The associated Cartan matrix is an $n \times n$ integer matrix whose entries are

$$c_{i,j} = \frac{2(\beta_i, \beta_j)}{(\beta_i, \beta_i)}.$$

Explicitely we get

$$\begin{bmatrix} 2 & -1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ -1 & 2 & -1 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & \ldots & \ldots & 0 \\ & & & \ddots & & & & \\ 0 & \ldots & \ldots & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \ldots & \ldots & -1 & 2 & -1 & -1 \\ 0 & 0 & \ldots & \ldots & 0 & -1 & 2 & 0 \\ 0 & 0 & \ldots & \ldots & 0 & -1 & 0 & 2 \end{bmatrix}.$$

Consider the root system B_n associated to the Lie algebra $\mathfrak{o}_{2n+1}$. Recall our choice of simple roots

$$\beta_k = e_k - e_{k-1},\ k = 1, \ldots, n-1, \quad \beta_n = e_n$$

from Exercise Sheet 9. The associated Cartan matrix is

$$\begin{bmatrix} 2 & -1 & 0 & \ldots & \ldots & \ldots & 0 \\ -1 & 2 & -1 & \ldots & \ldots & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & \ldots & 0 \\ & & & \ddots & & & \\ 0 & \ldots & \ldots & -1 & 2 & -1 & 0 \\ 0 & 0 & \ldots & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & \ldots & 0 & -2 & 2 \end{bmatrix}.$$

Consider the root system C_n associated to the Lie algebra $\mathfrak{sp}_{2n}$. Recall our choice of simple roots

$$\beta_k = e_k - e_{k-1},\ k = 1, \ldots, n-1, \quad \beta_n = 2e_n$$

from Exercise Sheet 9. The associated Cartan matrix is

$$\begin{bmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & \dots & 0 \\ & & & \ddots & & & \\ 0 & \dots & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & \dots & -1 & 2 & -2 \\ 0 & 0 & \dots & \dots & 0 & -1 & 2 \end{bmatrix}.$$

2. For any root system $R \subset U$, show that

$$R^\vee = \left\{ \alpha^\vee(-) = \frac{2(-,\alpha)}{(\alpha,\alpha)} \Big| \alpha \in R \right\} \subset U^*$$

is also a root system (we use the fact that the inner product identifies $U \cong U^*$). Show that if $\alpha_1, \dots, \alpha_r$ is a set of simple roots of R, then $\alpha_1^\vee, \dots, \alpha_r^\vee$ is a set of simple roots of $R^\vee$.

Solution. We make the usual identification

$$U \xrightarrow{\cong} U^*, \quad x \mapsto (-,x), \tag{1}$$

which induces a scalar product on U^*. Let α, β be two roots in R. Then $(\alpha^\vee, \alpha^\vee) = \frac{4}{(\alpha,\alpha)}$, and

$$2\frac{(\alpha^\vee, \beta^\vee)}{(\alpha^\vee, \alpha^\vee)} = 2\frac{(\alpha,\alpha)}{4}\frac{2 \cdot 2(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)} = 2\frac{(\alpha,\beta)}{(\beta,\beta)} \tag{2}$$

is an integer. Moreover, we observe that $\alpha^\vee$ is a multiple of $(-,\alpha)$, so the reflection $s_{\alpha^\vee}$ is the same as $s_{(-,\alpha)}$. Thus,

$$s_{\alpha^\vee}(\beta^\vee) = s_{(-,\alpha)}\left(\frac{2(-,\beta)}{(\beta,\beta)}\right) = s_{(-,\alpha)}\left(\frac{2(-,\beta)}{(s_\alpha(\beta), s_\alpha(\beta))}\right) = (s_\alpha(\beta))^\vee, \tag{3}$$

where we also used the fact that s_α is an isometry. Equations (2) and (3) imply that $R^\vee$ is again a root system.

The choice of a hyperplane $V \subset U$ determines positive roots $R^+ \subset R$. The associated simple roots $\alpha_1, \dots, \alpha_r$ are the indecomposable ones in R^+. Clearly the map (1) identifies V with a hyperplane $V^* \subset U^*$, and the set of positive roots determined by V^* is exactly the image $(R^+)^\vee$. We have to show that the set of indecomposable roots in $(R^+)^\vee$ is exactly $\alpha_1^\vee, \dots, \alpha_r^\vee$. We make use of the following Lemma, which is not difficult to prove.

Lemma 1. *For any root system $R \subset U$, let R^+ be the positive roots, and let α be a simple root. Then α cannot be written as a linear combination of elements in $R^+ - \{\alpha\}$ with non-negative coefficients.*

In our case let α be a positive root of R, but not simple. Then $\alpha^\vee$ is a linear combination of simple roots in $(R^+)^\vee$ with non-negative coeeficients. Then, Lemma 1 tells us that $\alpha^\vee$ cannot be a simple root, which implies our claim.

3. Show that the Weyl group of an irreducible root system acts transitively on the set of roots of given length (i.e. for any pair of short/long roots, there is a Weyl group element sending one to the other). *Hint: use the fact that $W \curvearrowright E$ is an irreducible representation (and prove this fact).*

Solution. Let α and β be two roots of the same length. We know from the proof of Proposition 27 in the Lecture Notes that $W \curvearrowright E$ is an irreducible representation and this allows us to find an element $w \in W$ such that $(w(\alpha), \beta) \neq 0$. Up to a reflection in W, we can also assume that the number $(w(\alpha), \beta)$ is positive, so that the angle θ between β and $w(\alpha)$ is acute. Since β and $w(\alpha)$ have the same length, we have $\cos(\theta) = \frac{c_{w(\alpha),\beta}}{2}$. Moreover, $c_{w(\alpha),\beta} \in \mathbb{Z}$. Summing up, we have

$$\cos(\theta) \in \{\frac{1}{2}, 1\}.$$

If the above number is 1 we are done, whereas if it is $\frac{1}{2}$, then the angle between β and $w(\alpha)$ is $\pi/3$. This means that the intersection of R with the plane generated by β and α contains A_2. Since the Weyl group of A_2 acts transitively, we are done.

4. Fill in the following gap in the proof of Theorem 19: if a Dynkin diagram consists of a triple edge connected to a double or triple edge, then the corresponding 3×3 matrix S (see Definition 23) is negative-definite.

Solution. We consider a Dynkin diagram consisting of a triple edge connected to a double edge and we compute its Cartan matrix C as follows.

$$C = \begin{bmatrix} 2 & c_{12} & 0 \\ c_{21} & 2 & c_{23} \\ 0 & c_{32} & 2 \end{bmatrix}, \tag{4}$$

where $c_{12}c_{21} = 3$, and $c_{23}c_{32} = 2$. We also have the diagonal matrix D from Definition 23, whose inverse is

$$D^{-1} = \begin{bmatrix} \frac{c_{1,1}}{2} & 0 & 0 \\ 0 & \frac{c_{2,2}}{2} & 0 \\ 0 & 0 & \frac{c_{3,3}}{2} \end{bmatrix}, \tag{5}$$

where $c_{i,i} > 0$ for $i = 1, 2, 3$. Then we compute the matrix S as

$$S = D^{-1}C = \begin{bmatrix} c_{1,1} & \frac{c_{1,1}c_{12}}{2} & 0 \\ \frac{c_{2,2}c_{2,1}}{2} & c_{2,2} & \frac{c_{2,2}c_{2,3}}{2} \\ 0 & \frac{c_{3,3}c_{3,2}}{2} & c_{3,3}. \end{bmatrix} \tag{6}$$

. We apply Sylvester‘s criterion exploiting the relations among the $c_{i,j}$‘s. The first two minors are $c_{1,1} > 0$ and $\frac{1}{4}c_{1,1}c_{2,2} > 0$. The third minor is

$$\frac{1}{4}c_{1,1}c_{2,2}c_{3,3} - \frac{c_{3,3}c_{3,2}}{2} \cdot c_{1,1}\frac{c_{2,2}c_{2,3}}{2} = -\frac{1}{4}c_{1,1}c_{2,2}c_{3,3} < 0.$$

The case of a Dynkin diagram consisting of a triple edge connected to a triple edge is analogous.

(*) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let G be the corresponding simply connected complex Lie group. For any root α (corresponding to a henceforth fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$), construct an element $S_\alpha \in G$ such that

$$\mathrm{Ad}_{S_\alpha} : \mathfrak{g}^* \to \mathfrak{g}^*$$

sends $\mathfrak{h}^*$ to $\mathfrak{h}^*$ and coincides with the simple reflection s_α. *Hint: deal first with the case* $\mathfrak{g} = \mathfrak{sl}_2$.